On a class of periodic quasilinear Schrödinger equations involving critical growth in $\mathbb{R}^2$

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Abstract

We consider the equation $-\Delta u + V(x)u - k(\Delta(|u|^2))u = g(x,u), u > 0, x \in \mathbb{R}^2$, where $V : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ are two continuous 1−periodic functions. Also, we assume g behaves like $\exp(\beta|u|^4)$ as $|u| \to \infty$. We prove the existence of at least one weak solution $u \in H^1(\mathbb{R}^2)$ with $u^2 \in H^1(\mathbb{R}^2)$. Mountain pass in a suitable Orlicz space together with Moser-Trudinger are employed to establish this result. Such equations arise when one seeks for standing wave solutions for the corresponding quasilinear Schrödinger equations. Schrödinger equations of this type have been studied as models of several physical phenomena. The nonlinearity here corresponds to the superfluid film equation in plasma physics.

Key words: Mountain pass, critical growth, standing waves, , quasilinear Schrödinger equations.
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1 Introduction

We are concerned with the existence of positive solutions for quasilinear elliptic equations in the entire space,

$$-\Delta u + V(x)u - k(\Delta(|u|^2))u = g(x,u), \quad u > 0, x \in \mathbb{R}^2,$$

where $V : \mathbb{R}^2 \to [0,\infty)$ and $g : \mathbb{R}^2 \times \mathbb{R} \to [0,\infty)$ are nonnegative continuous functions. Solutions of this equation are related to the existence of standing wave solutions for quasilinear Schrödinger

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equations of the form

$$i\partial_t z = -\Delta z + W(x)z - l(x,|z|)z - k\Delta h(|z|^2)h'(|z|^2)z, \quad x \in \mathbb{R}^N, N \geq 2, \tag{1}$$

where $W(x)$ is a given potential, k is a real constant and l and h are real functions. Quasilinear equations of the form (1) have been established in several areas of physics corresponding to various types of h. The superfluid film equation in plasma physics has this structure for $h(s) = s$, (Kurihura in [8]). In the case $h(s) = (1+s)^{1/2}$, equation (1) models the self-channeling of a high-power ultra short laser in matter, see [21]. Equation (1) also appears in fluid mechanics [8,9], in the theory of Heidelberg ferromagnetism and magnus [10], in dissipative quantum mechanics and in condensed matter theory [14]. We consider the case $h(s) = s$ and $k > 0$. Setting $z(t,x) = \exp(-iFt)u(x)$ one obtains a corresponding equation of elliptic type which has the formal variational structure:

$$-\Delta u + V(x)u - k(\Delta(|u|^2))u = g(x,u), \quad u > 0, x \in \mathbb{R}^N, \tag{2}$$

where $V(x) = W(x) - F$ is the new potential function and g is the new nonlinearity.

Note that, for the case $g(u) = |u|^{p-1}u$ with $N \geq 3$, $p+1 = 2(2^*) = \frac{4N}{N-2}$ behaves like a critical exponent for the above equation [13, Remark 3.13]. For the subcritical case $p+1 < 22^*$ the existence of solutions for problem (2) was studied in [10, 11, 12, 14, 15, 16] and it was left open for the critical exponent case $p+1 = 2(2^*)$ [13; Remark 3.13]. The author in [16], proved the existence of solutions for $p+1 = 2(2^*)$ with $N \geq 3$ whenever the potential function $V(x)$ satisfies some geometry conditions. It the present paper, we will extend this result for the case $N = 2$. It is well-known that for the semilinear case ($k = 0$),

$$-\Delta u + V(x)u = g(u), \quad u > 0, x \in \mathbb{R}^N, \tag{P}$$

$p+1 = 2^*$ is the critical exponent when $N \geq 3$. Here is the definition of the critical growth for $N = 2$,

- *Critical growth:* There exists $\beta_0 > 0$ such that

$$\lim_{t\to+\infty} \frac{|g(x,t)|}{\exp(\beta t^2)} = 0 \qquad \forall \beta > \beta_0, \quad \lim_{t\to+\infty} \frac{|g(x,t)|}{\exp(\beta t^2)} = +\infty \qquad \forall \beta < \beta_0$$

uniformly with respect to $x \in \mathbb{R}^2$. Note that the corresponding critical growth for $N = 2$ comes from a version of Trudinger-Moser inequality in the whole space $\mathbb{R}^2$ (see [6]) as follows,

$$\int_{\mathbb{R}^2} \big(\exp(\beta|u|^2) - 1\big)\, dx < +\infty, \qquad \forall u \in H^1(\mathbb{R}^2), \beta > 0.$$

Also, if $\beta < 4\pi$ and $|u|_{L^2(\mathbb{R}^2)} \leq C$, there exists a constant $C_2 = C_2(C,\beta)$ such that

$$\sup_{|\nabla u|_{L^2(\mathbb{R}^2)}\leq 1} \int_{\mathbb{R}^2} \big(\exp(\beta|u|^2) - 1\big)\, dx < C_2.$$

There are many results about the existence of solutions for the subcritical, critical and the supercritical exponent case for problem (P)(e.g. [1, 3, 4, 5, 19, 22]).

In the case $k > 0$, for the subcritical case, the existence of a nonnegative solution was proved for $N = 1$ by Poppenberg, Schmitt and Wang in [18] and for $N \geq 2$ by Liu and Wang in [12]. In [13] Liu and Wang improved these results by using a change of variables and treating the new problem in an Orlicz space. The author in [15], using the idea of the fibrering method, studied this problem in connection with the corresponding eigenvalue problem for the laplacian $-\Delta u = V(x)u$ and proved the existence of multiple solutions for problem (2). It is established in [11], the existence of both one-sign and nodal ground states of soliton type solutions by the Nehari method. They also established some regularity of the positive solutions.
As it was mentioned above, for the case $k > 0$ with $g(u) = |u|^{p-1}u$ and $N \geq 3$, $p + 1 = 2(2^*) = \frac{4N}{N-2}$ behaves like a critical exponent for problem (2). This is because of the nonlinearity term $-\epsilon k(\Delta(|u|^2))u$. Therefore for problem (2), because of the presence of this nonlinearity term, the above definition of Critical growth for $N = 2$ changes as follows:

- ***Critical growth:*** If $N = 2$, there exists $\beta_0 > 0$ such that

$$\lim_{t\to+\infty} \frac{|g(x,t)|}{\exp(\beta t^4)} = 0 \qquad \forall \beta > \beta_0, \quad \lim_{t\to+\infty} \frac{|g(x,t)|}{\exp(\beta t^4)} = +\infty \qquad \forall \beta < \beta_0,$$

uniformly with respect to $x \in \mathbb{R}^2$
Here, we shall study problem (2) with $N = 2$ and show the existence of positive solutions when the function g has the critical growth. Before to state the main result, we fix the hypotheses on the potential function V and the function g. Indeed,

H1: $V : \mathbb{R}^2 \to \mathbb{R}$ is a continuous 1−periodic function satisfying $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^2$.

H2: $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous 1−periodic function satisfying $g(x, s) = o_1(s)$ near origin uniformly with respect to $x \in \mathbb{R}^2$.

H3: g has critical growth at $+\infty$, namely,

$$g(x, s) \leq C(e^{4\pi s^4} - 1) \text{ for all } (x, s) \in \mathbb{R}^2 \times [0, +\infty).$$

H4: The Ambrosetti-Rabinowitz growth condition: There exists $\theta > 4$ such that

$$0 \leq \theta G(x,t) = \theta \int_0^t g(x,s)\,ds \leq tg(x,t), \qquad t > 0.$$

H5: For each fixed $x \in \mathbb{R}^2$, the function $\frac{g(x,s)}{s}$ is increasing with respect to s, for $s > 0$.

H6: There are constants $p > 2$ and C_p such that

$$g(x,s) \geq C_p s^{p-1} \text{ for all } (x,s) \in \mathbb{R}^2 \times [0,+\infty),$$

where

$$C_p \quad > \quad \Big[\frac{\theta(p-2)}{p(\theta-4)}\Big]^{(p-2)/2} S_p^p, \tag{3}$$

$$S_p \quad := \quad \inf_{u \in H_r^1(\mathbb{R}^2)\setminus\{0\}} \frac{\big(\int_{\mathbb{R}^2}(|\nabla u|^2 + V_1 u^2)\,dx + (\int_{\mathbb{R}^2} u^2|\nabla u|^2\,dx)^{1/2}\big)^{1/2}}{\big(\int_{\mathbb{R}^2}|u|^p\,dx\big)^{1/p}}, \tag{4}$$

and $V_1 := \max_{x\in\mathbb{R}^2} V(x)$.

It follows from Theorem 1.1 in [12] by some obvious changes that the infimum in (4) attains. Here is our main Theorem.

Theorem 1.1. *Assume Conditions $H1 - H6$. Then,* (2) *possesses a nontrivial weak solution $\tilde{u} \in H^1(\mathbb{R}^N)$ with $\tilde{u}^2 \in H^1(\mathbb{R}^N)$.*

This paper is organized as follows. In Section 2, we reformulate this problem in an appropriate Orlicz space. Theorem 1.1 is proved in Section 3.

2 Reformulation of the problem and preliminaries

In this section we assume $N \geq 2$. Denote by $H_r^1(\mathbb{R}^N)$ the space of radially symmetric functions in

$$H^{1,2}(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\right\}.$$

Without loss of generality, one can assume $k = 1$ in problem (2). We formally formulate problem (2) in a variational structure as follows

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1+u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} G(x,u)dx.$$

on the space

$$X = \big\{u \in H^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\big\},$$

which is equipped with the following norm,

$$\|u\|_X = \left\{\int_{\mathbb{R}^N}|\nabla u|^2 dx + \int_{\mathbb{R}^N} V(x)u^2 dx\right\}^{\frac{1}{2}}.$$

Liu and Wang in [13] for the subcritical case, by making a change of variables treated this problem in an Orlicz space. Following their work, we consider this problem for the supercritical exponent case in the same Orlicz space. To convince the reader we briefly recall some of their notations and results that are useful in the sequel.

First, we make a change of variables as follows,

$$dv = \sqrt{1+u^2}du, \quad v = h(u) = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2}).$$

Since h is strictly monotone it has a well-defined inverse function: $u = f(v)$. Note that

$$h(u) \sim \begin{cases} u, & |u| \ll 1 \\ \frac{1}{2}u|u|, & |u| \gg 1, \quad h'(u) = \sqrt{1+u^2}, \end{cases}$$

and

$$f(v) \sim \begin{cases} v & |v| \ll 1 \\ \sqrt{\frac{2}{|v|}}v, & |v| \gg 1, \quad f'(v) = \frac{1}{h'(u)} = \frac{1}{\sqrt{1+u^2}} = \frac{1}{\sqrt{1+f^2(v)}}. \end{cases}$$

Also, for some $C_0 > 0$ it holds

$$L(v) := f(v)^2 \sim \begin{cases} v^2 & |v| \ll 1, \\ 2|v| & |v| \gg 1, \quad L(2v) \leq C_0 L(v), \end{cases}$$

$L(v)$ is convex, $L'(v) = \frac{2f(v)}{\sqrt{1+f(v)^2}}$, $L''(v) = \frac{2}{(1+f(v)^2)^2} > 0$.
Using this change of variable, we can rewrite the functional $J(u)$ as

$$\bar{J}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f(v)^2 dx - \int_{\mathbb{R}^N} G(x, f(v)) dx.$$

$\bar{J}$ is defined on the space

$$H^1_L(\mathbb{R}^N) = \big\{v | v(x) = v(|x|), \int_{\mathbb{R}^N} |\nabla v|^2 dx < \infty, \int_{\mathbb{R}^N} V(x) L(v) dx < \infty\big\}.$$

We introduced the Orlicz space (e.g.[20])

$$E_L(\mathbb{R}^N) = \big\{v | \int_{\mathbb{R}^N} V(x) L(v) dx < \infty\big\},$$

equipped with the norm

$$|v|_{E_L(\mathbb{R}^N)} = \inf_{\zeta > 0} \zeta\big(1 + \int_{\mathbb{R}^N} (V(x) L(\zeta^{-1} v(x)) dx\big),$$

and define the norm of $H^1_L(\mathbb{R}^N)$ by

$$\|v\|_{H^1_L(\mathbb{R}^N)} = |\nabla v|_{L^2(\mathbb{R}^N)} + |v|_{E_L(\mathbb{R}^N)}.$$

Here are some related facts.

Proposition 2.1. *(i) $E_L(\mathbb{R}^N)$ is a Banach space.*

(ii) If $v_n \longrightarrow v$ in $E_L(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} V(x)|L(v_n) - L(v)| dx \longrightarrow 0$ and $\int_{\mathbb{R}^N} V(x)|f(v_n) - f(v)|^2 dx \longrightarrow 0$.

(iii) *If* $v_n \longrightarrow v$ *a.e. and* $\int_{\mathbb{R}^N} V(x)L(v_n)dx \longrightarrow \int_{\mathbb{R}^N} V(x)L(v)dx$*, then* $v_n \longrightarrow v$ *in* $E_L(\mathbb{R}^N)$*.*

(iv) *The dual space* $E_L^*(\mathbb{R}^N) = L^\infty \cap L_V^2 = \{w | w \in L^\infty, \int_{\mathbb{R}^N} V(x)w^2 dx < \infty\}$*.*

(v) *If* $v \in E_L(\mathbb{R}^N)$*, then* $w = L'(v) = 2f(v)f'(v) \in E_L^*(\mathbb{R}^N)$*, and* $|w|_{E_L^*} = \sup_{|\phi|_{E_L(\mathbb{R}^N)} \leq 1}(w, \phi) \leq C_1\big(1 + \int_{\mathbb{R}^N} V(x)L(v)dx\big)$*, where* C_1 *is a constant independent of* v*.*

(vi) *For* $N > 2$ *the map:*$v \longrightarrow f(v)$ *from* $H_L^1(\mathbb{R}^N)$ *into* $L^q(\mathbb{R}^N)$ *is continuous for* $2 \leq q \leq 22^*$ *and is compact for* $2 < q < 22^*$*. Also, for* $N = 2$*, this map is continuous for* $q > 1$*.*

(vii) *Suppose* $0 < V_0 \leq V(x) < V_1$*. There exists a positive constant* C *such that*

$$\|u^2\|_{H^{1,2}(\mathbb{R}^N)} \leq C(\|v\|_{H_L^1(\mathbb{R}^N)} + \|v\|^2_{H_L^1(\mathbb{R}^N)}),$$

where $u = f(v)$*.*

Proof. See Propositions (2.1) and (2.2) in [13] for the proof of parts (i) to (vi). We prove part (vii). A direct computation shows that

$$\int |\nabla u^2|^2 \, dx = 4 \int \frac{f(v)^2 |\nabla v|^2}{1 + f(v)^2} \, dx \leq 4\|v\|^2_{H_L^1(\mathbb{R}^N)}.$$

Also, from part (vi) we have

$$\int u^4 \, dx = \int f(v)^4 \, dx \leq C\|v\|^4_{H_L^1(\mathbb{R}^N)}.$$

Now the result is deduced from the above inequalities. □

Hence forth, $\int, H^1, H_r^1, H_L^1, E_L, L^t, |\cdot|_L$ and $\|\cdot\|$ stand for $\int_{\mathbb{R}^2}$, $H^{1,2}(\mathbb{R}^2)$, $H_r^1(\mathbb{R}^2)$, $H_L^1(\mathbb{R}^2)$, $E_L(\mathbb{R}^2)$, $L^t(\mathbb{R}^2)$, $|\cdot|_{E_L(\mathbb{R}^2)}$ and $\|\cdot\|_{H_L^1(\mathbb{R}^2)}$ respectively. In the following we use C to denote any constant that is independent of the sequences considered.

3 Proof of Theorem 1.1

In this section, we combine the arguments used in [1] and [16] to prove Theorem 1.1. The following proposition states some properties of the functional $\bar{J}$.

Proposition 3.1. *(i)* $\bar{J}$ *is well-defined on* H_L^1*.*

(ii) $\bar{J}$ *is continuous in* H_L^1*.*

(iii) $\bar{J}$ *is Gauteaux-differentiable in* H_L^1*.*

Proof. The proof is similar to the proof of Proposition (2.3) in [13] by some obvious changes.□

Here, we prove the existence of a critical point for the functional $\bar{J}$.

Theorem 3.2. *$\bar{J}$ has a critical point in H_L^1, that is, there exists $0 \neq v \in H_L^1$ such that*

$$\int \nabla v . \nabla \phi dx + \int V(x) f(v) f'(v) \phi dx - \int g(x, f(v)) f'(v) \phi dx = 0,$$

for every $\phi \in H_L^1$.

We use the Mountain Pass Theorem (see [2], [19]) to prove Theorem 3.2. First, let us define the Mountain Pass value,

$$C_0 := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \bar{J}(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], H_L^1) | \gamma(0) = 0, \bar{J}(\gamma(1)) \leq 0, \gamma(1) \neq 0\}.$$

The following lemmas are crucial for the proof of Theorem 3.2.

Lemma 3.3. *The functional $\bar{J}$ satisfies the Mountain Pass Geometry.*

Proof. We need to show that there exists $0 \neq v \in H_L^1$ such that $\bar{J}(v) \leq 0$. Let $0 \neq u \in C_0^\infty(\mathbb{R}^2)$. It is easy to see that $J(tu) \leq 0$ for the large values of t. Consequently $\bar{J}(v) < 0$ where $v = h(tu)$. □

Lemma 3.4. C_0 *is positive.*

Proof. Set

$$S_\rho := \{v \in H_L^1 | \int |\nabla v|^2 dx + \int V(x) f(v)^2 dx = \rho^2\}.$$

It follows from $H2$ that there exists $\delta > 0$ such that

$$G(x, s) \leq \frac{|s|^2}{4V_1}, \qquad |s| \leq \delta. \tag{5}$$

Also, for $|s| > \delta$,it follows from $H3$ and $H4$ that

$$\delta^p G(x, s) \leq |s|^p G(x, s) \leq \frac{|s|^{p+1} g(x, s)}{\theta} \leq C|s|^{p+1}(e^{4\pi s^2} - 1). \tag{6}$$

By (5) and (6), we get

$$G(x, s) \leq \frac{|s|^2}{4V_1} + \frac{C}{\delta^p}|s|^{p+1}(e^{4\pi s^2} - 1). \tag{7}$$

Set $u = f(v)$, with $v \in S_\rho$ and $\rho < 1$. By part (vii) of Proposition 2.1

$$\|u^2\|_{H^{1,2}(\mathbb{R}^N)} \leq C(\|v\|_{H_L^1(\mathbb{R}^N)} + \|v\|^2_{H_L^1(\mathbb{R}^N)}),$$

hence, if ρ is sufficiently small, it follows from Trudinger-Moser inequality and the above inequality that

$$\int (e^{4q\pi f(v)^4} - 1)\, dx = \int (e^{4q\pi u^4} - 1)\, dx \le C$$

for every $q > 1$ close to one. Therefore, it follows from the above inequality and Hölder inequality that, $(\frac{1}{q} + \frac{1}{q'} = 1)$

$$\begin{aligned} \int |f(v)|^{p+1}(e^{4\pi f(v)^4} - 1)\, dx &\le \Big(\int |f(v)|^{q'(p+1)}\, dx\Big)^{\frac{1}{q'}}\Big(\int (e^{4q\pi f(v)^4} - 1)\, dx\Big)^{\frac{1}{q}} \\ &\le C\Big(\int |f(v)|^{q'(p+1)}\, dx\Big)^{\frac{1}{q'}} \\ &\le C\|v\|^{p+1} \end{aligned} \tag{8}$$

Taking into account (7) and (8) for each $v \in S_\rho$ with $\rho << 1$, we have

$$\int G(x, f(v))\, dx \le \frac{1}{4}\rho^2 + \frac{C}{\delta^p}\rho^{p+1} \tag{9}$$

Considering (9) and the fact that $v \in S_\rho$, we obtain

$$\begin{aligned} \bar{J}(v) &= \frac{1}{2}\int |\nabla v|^2 dx + \frac{1}{2}\int V(x) f(v)^2 dx - \int G(x, f(v)) dx \\ &\ge \frac{1}{2}\rho^2 - \frac{1}{4}\rho^2 - \frac{C}{\delta^P}\rho^{p+1} \ge \frac{1}{8}\rho^2, \end{aligned}$$

when $0 < \rho \le \rho_0 \ll 1$ for some ρ_0. Hence, for $v \in S_\rho$ with $0 < \rho \le \rho_0$ we have

$$\bar{J}(v) \ge \frac{1}{8}\rho^2. \tag{10}$$

If $\gamma(1) = v$ and $\bar{J}(\gamma(1)) < 0$ then it follows from (10) that

$$\int |\nabla v|^2 dx + \int V(x) f(v)^2 dx > \rho_0^2,$$

thereby giving

$$\sup_{t\in[0,1]} \bar{J}(\gamma(t)) \ge \sup_{\gamma(t)\in S_{\rho_0}} \bar{J}(\gamma(t)) \ge \frac{1}{8}\rho_0^2.$$

Therefore $C_0 \ge \frac{1}{8}\rho_0^2 > 0.\square$

Lemma 3.5. *C_0 is bounded from above by $\frac{\theta-4}{2\theta}$.*

Proof. We fix a positive radial function $\phi \in H_r^1$ such that

$$S_p = \frac{\big(\int_{\mathbb{R}^2}(|\nabla\phi|^2 + V_1\phi^2)\, dx + (\int_{\mathbb{R}^2}\phi^2|\nabla\phi|^2\, dx)^{1/2}\big)^{1/2}}{\big(\int_{\mathbb{R}^2}|\phi|^p\, dx\big)^{1/p}}, \quad \text{and } \int_{\mathbb{R}^2}\phi^2|\nabla\phi|^2\, dx \le 1.$$

Set $\gamma_1(t) := h(t\phi)$. It follows from the definition of the Mountain Pass value that

$$C_0 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \bar{J}(\gamma(t)) \le \sup_{t\in[0,1]} \bar{J}(\gamma_1(t)) = \sup_{t\in[0,1]} \bar{J}(h(t\phi)) = \sup_{t\in[0,1]} J(t\phi).$$

Therefore, we obtain

$$\begin{aligned} C_0 &\le \sup_{t\in[0,1]} J(t\phi) \\ &\le \sup_{t\in[0,1]} \frac{t^2}{2}\int (|\nabla\phi|^2 + V_1\phi^2)\,dx + \frac{t^4}{2}\int \phi^2|\nabla\phi|^2\,dx - \int G(x,t\phi)dx \\ &\le \sup_{t\in[0,1]} \frac{t^2}{2}\int (|\nabla\phi|^2 + V_1\phi^2)\,dx + \frac{t^2}{2}\Big(\int \phi^2|\nabla\phi|^2\,dx\Big)^{1/2} - \int G(x,t\phi)dx \\ &\le \sup_{t\in[0,1]} \frac{t^2}{2}\int (|\nabla\phi|^2 + V_1\phi^2)\,dx + \frac{t^2}{2}\Big(\int \phi^2|\nabla\phi|^2\,dx\Big)^{1/2} - t^pC_p\int \phi^p dx \\ &= \frac{(p-2)S_p^{\frac{2p}{p-2}}}{2pC_p^{\frac{2}{p-2}}}. \end{aligned}$$

Also, it follows from $H6$ that $\frac{(p-2)S_p^{\frac{2p}{p-2}}}{2pC_p^{\frac{2}{p-2}}} < \frac{\theta-4}{2\theta}$ which implies $C_0 < \frac{\theta-4}{2\theta}$. $\square$

The Mountain Pass Theorem guaranties the existence of a $(PS)_{C_0}$ sequence $\{v_n\}$, that is, $\bar{J}(v_n) \longrightarrow C_0$ and $\bar{J}'(v_n) \longrightarrow 0$. The following lemma states some properties of this sequence.

Lemma 3.6. *Suppose $\{v_n\}$ is a $(PS)_{C_0}$ sequence. The following statements hold.*

(i) $\{v_n\}$ is bounded in H^1_L.

(ii) If $v_n \ge 0$ converges weakly to v in H^1_L, then for every nonnegative test function $\phi \in H^1_L$ we have

$$\lim_{n\to+\infty} \langle \bar{J}'(v_n), \phi\rangle = \langle \bar{J}'(v), \phi\rangle.$$

Proof. Since $\{v_n\}$ is a $(PS)_{C_0}$ sequence, we have

$$\begin{aligned} \bar{J}(v_n) &= \frac{1}{2}\int |\nabla v_n|^2 dx + \frac{1}{2}\int V(x)f(v_n)^2dx - \int G(x,f(v_n))dx \\ &= C_0 + o(1), \end{aligned} \tag{11}$$

and

$$\begin{aligned} \langle \bar{J}'(v_n), \phi\rangle &= \int \nabla v_n.\nabla\phi dx + \int V(x)f(v_n)f'(v_n)\phi dx - \int g(x,f(v_n))f'(v_n)\phi dx \\ &= o(\|\phi\|) \end{aligned} \tag{12}$$

For part (i), pick $\phi = \frac{f(v_n)}{f'(v_n)} = \sqrt{1+f(v_n)^2}f(v_n)$ as a test function. One can easily deduce that $|\phi|_L \le C|v_n|_L$ and

$$|\nabla\phi| = (1 + \frac{f(v_n)^2}{1+f(v_n)^2})|\nabla v_n| \le 2|\nabla v_n|,$$

which implies $\|\phi\| \le C\|v_n\|$. Substituting ϕ in (8), gives

$$\begin{aligned}\langle \bar{J}'(v_n), \frac{f(v_n)}{f'(v_n)}\rangle = &\int (1 + \frac{f(v_n)^2}{1+f(v_n)^2})|\nabla v_n|^2 dx + \int V(x) f(v_n)^2 dx \\ &- \int g(x, f(v_n)) f(v_n) dx \\ =& \, o(\|v_n\|). \end{aligned} \tag{13}$$

Taking into account (11), (12) and (13), we have

$$\begin{aligned} C_0 + o(1) + o(\|v_n\|) =& \bar{J}(v_n) - \frac{1}{\theta}\langle \bar{J}'(v_n), \frac{f(v_n)}{f'(v_n)}\rangle \\ =& \frac{1}{2}\int |\nabla v_n|^2 dx + \frac{1}{2}\int V(x) f(v_n)^2 dx \\ &- \frac{1}{\theta}\int (1 + \frac{f(v_n)^2}{1+f(v_n)^2})|\nabla v_n|^2 dx - \frac{1}{\theta}\int V(x) f(v_n)^2 dx \\ =& \int (\frac{1}{2} - \frac{1}{\theta}(1 + \frac{f(v_n)^2}{1+f(v_n)^2}))|\nabla v_n|^2 dx + (\frac{1}{2} - \frac{1}{\theta})\int V(x) f(v_n)^2 dx \\ \ge& (\frac{1}{2} - \frac{2}{\theta})\int |\nabla v_n|^2 dx + (\frac{1}{2} - \frac{1}{\theta})\int V(x) f(v_n)^2 dx \\ \ge& \frac{(\theta - 4)}{2\theta}\int \left(|\nabla v_n|^2 + V(x) f(v_n)^2\right) dx. \end{aligned}$$

Since $\theta > 4$ it follows from the above that $\int |\nabla v_n|^2 dx + \int V(x) f(v_n)^2 dx$ is bounded and indeed,

$$\limsup_{n\to\infty} \|v_n\|^2 := K \le \frac{2\theta C_0}{\theta - 4}. \tag{14}$$

It proves part (i).

For part (ii), note first that it follows from (14) and Lemma (3.5) that

$$\limsup_{n\to\infty} \|v_n\|^2 = K < 1.$$

From Trudinger-Moser inequality, there exists $\gamma, q > 1$ sufficiently close to one that $T_n(x) := e^{4\pi f(v_n)^2} - 1$ is bounded in L^q. Since, $v_n \to v$ a.e. in $\mathbb{R}^2$ so $T_n(x) \rightharpoonup T(x) = e^{4\pi f(v)^2} - 1$ weakly in L^q. Now for each $\phi \in H_L^1$ since $H_L^1 \subseteq L^t$ for $t > 1$ (Proposition 2.1 part (vi)) we have

$$\int T_n(x)\phi\, dx \to \int T(x)\phi\, dx.$$

Since, f is increasing and $f(0) = 0$, hence $f(v_n) \ge 0$ and $f(v) \ge 0$. Now it follows from H_3 that

$$g(x, f(v_n)) f'(v_n)\phi \le C T_n(x)\phi.$$

Hence, the dominated convergence theorem implies

$$\int g(x, f(v_n)) f'(v_n)\phi\, dx \to \int g(x, f(v)) f'(v)\phi\, dx. \tag{15}$$

For the second term on the right hand side of (12), we have

$$V(x)f(v_n)f'(v_n)\phi \leq V(x)f(v_n)\phi,$$

and since $v_n \rightharpoonup v$ weakly in H_1^G, for the right hand side of the above inequality we have

$$\lim_{n\to\infty}\int V(x)f(v_n)\phi\,dx = \int V(x)f(v)\phi\,dx.$$

Hence by the dominated convergence theorem and the fact that $v_n \to v$ a.e. we obtain

$$\lim_{n\to\infty}\int V(x)f(v_n)f'(v_n)\phi\,dx = \int V(x)f(v)f'(v)\phi\,dx. \tag{16}$$

It follows from (12), (15) and (16) that

$$\lim_{n\to+\infty}\langle \bar{J}'(v_n),\phi\rangle = \langle \bar{J}'(v),\phi\rangle.$$

It proves part (ii). □

Here is a version of Lions' results applicable in our setting.

Lemma 3.7. *Suppose $v_n \to 0$ in H_L^1. If there exists $R > 0$ such that*

$$\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_R(y)}|f(v_n)|^2\,dx = 0, \tag{17}$$

then,

$$\int f(v_n)g(x,f(v_n))\,dx \to 0, \ \text{ as } n\to\infty.$$

Proof. It follows from (17) and Lemma (4.8) in [7] that

$$f(v_n)\to 0 \text{ in } L^t \text{ for all } t\in(2,+\infty). \tag{18}$$

Also, since $\limsup_{n\to\infty}\|v_n\|^2 = K < 1$, it follows from Trudinger-Moser inequality that

$$\int (e^{4\gamma\pi f(v_n)^2}-1)\,dx \leq C,$$

for $\gamma > 1$ sufficiently close to one. Now, by the same argument to prove the inequality (7), for any $\epsilon > 0$ there exist constants C_ϵ and $q,\gamma > 1$ sufficiently close to one that

$$\begin{aligned}
\int f(v_n)g(x,f(v_n))\,dx &\leq \epsilon\int f(v_n)^2\,dx + C_\epsilon\int f(v_n)(e^{4\gamma\pi f(v_n)^2}-1)\,dx\\
&\leq \epsilon C + C_\epsilon\Big(\int |f(v_n)|^{q'}\,dx\Big)^{\frac{1}{q'}}\Big(\int (e^{4\gamma q\pi f(v_n)^2}-1)\,dx\Big)^{\frac{1}{q}}\\
&\leq \epsilon C + C_\epsilon\Big(\int |f(v_n)|^{q'}\,dx\Big)^{\frac{1}{q'}}.
\end{aligned}$$

From the above inequality together with (18), we obtain

$$\int f(v_n)g(x,f(v_n))\,dx \to 0, \text{ as } n\to\infty.$$

□

Lemma 3.8. *There exist a sequence* (y_n) *in* $\mathbb{R}^2$ *and* $R, \epsilon > 0$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |f(v_n)|^2\, dx > \epsilon,$$

Proof. If $\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} |f(v_n)|^2\, dx = 0$, it follows from Lemma (3.7) that

$$\int f(v_n) g(x, f(v_n))\, dx \to 0, \text{ as } n \to \infty.$$

This implies $C_0 = 0$ and it is a contradiction by virtue of Lemma (3.4).□

Proof of Theorem 3.2. It follows from Lemma (3.8) the existence of a sequence (y_n) in $\mathbb{R}^2$ such that the result of Lemma (3.7) holds for some $R, \epsilon > 0$. Without loss of generality, one can assume $(y_n) \subset \mathbb{Z}^2$. Now let $\tilde{v_n} = v_n(x - y_n)$. Since, $V(.), g(., s)$ and $G(., s)$ are $1-$periodic we have

$$\|v_n\| = \|\tilde{v_n}\|, J(v_n) = J(\tilde{v_n}) \text{ and also } J'(\tilde{v_n}) \to 0.$$

Since $\|\tilde{v_n}\|$ is bounded, there exists $\tilde{v_0} \in H^1_L$ such that $\tilde{v_n} \to \tilde{v_0}$ in H^1_L. Now, it follows from Lemma (3.6) that $J'(\tilde{v_0}) = 0$. Also, by Lemma (3.8) we have

$$\epsilon \le \int_{B_R(0)} |f(\tilde{v_n})|^2\, dx < 2\int_{B_R(0)} |f(\tilde{v_n}) - f(\tilde{v_0})|^2\, dx + 2\int_{B_R(0)} |f(\tilde{v_0})|^2\, dx,$$

which implies $\tilde{v_0} \not\equiv 0$. □

Proof of Theorem 1.1. Proof is a direct consequence of Theorem (3.2). Indeed, since $\tilde{v_0} \not\equiv 0$ is a critical point of $\bar{J}$, it is easily seen that $\tilde{u} = f(\tilde{v_0})$ is a nontrivial critical point of J. □

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